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Exact S-matrix for N-disc systems and various boundary conditions: II. Determination and partial classification of resonances

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Abstract. Scattering of waves and particles from two- and three-disc systems with discrete $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ symmetries is studied for various boundary conditions. Resonances are numerically determined, and partially classified by using the irreducible representations of the symmetry groups. New physical effects are expected (splitting up of resonances and resonances of interaction between the scatterers). Some of these effects can be observed on the far-field form functions.

1. Introduction

In a previous paper [1], we have developed an exact formalism allowing us to calculate the S-matrix, its scattering resonances and far-field form functions for systems with discrete symmetries. Various boundary conditions on the scatterers, corresponding to mesoscopic quantum physics, acoustics or electromagnetism, have been considered.

In this second part of our work, we are more particularly concerned with the determination of the resonances of the two- and three-disc systems with discrete $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ symmetries. In section 2, the scattering resonances (poles of the S-matrix) are numerically determined and partially classified for various configurations. Some interesting effects, such as splitting up of resonances and resonances of interaction between the scatterers, can be directly observed on the corresponding far-field form functions. In section 3, experimental and theoretical extensions of our work are considered.

2. Study of scattering resonances and discussion

2.1. Symmetry $\mathcal{C}_{2v}$

We present in figures 1–4 the locations of the resonances for Dirichlet, Neumann, mixed and elastic boundary conditions. In all cases, the centre-to-centre separation $d = 6a$. In the case of mixed boundary conditions, the refraction index $n = 1.1$. In the case of elastic boundary conditions, tungsten carbide cylinders immersed in water are considered. The computations were carried out for the following parameters: water ($\rho = 1$ g cm^{-3}, $c = 1480$ m s^{-1}), and tungsten carbide ($\rho' = 13.80$ g cm^{-3}, $c_L = 6860$ m s^{-1}, $c_T = 4185$ m s^{-1}).

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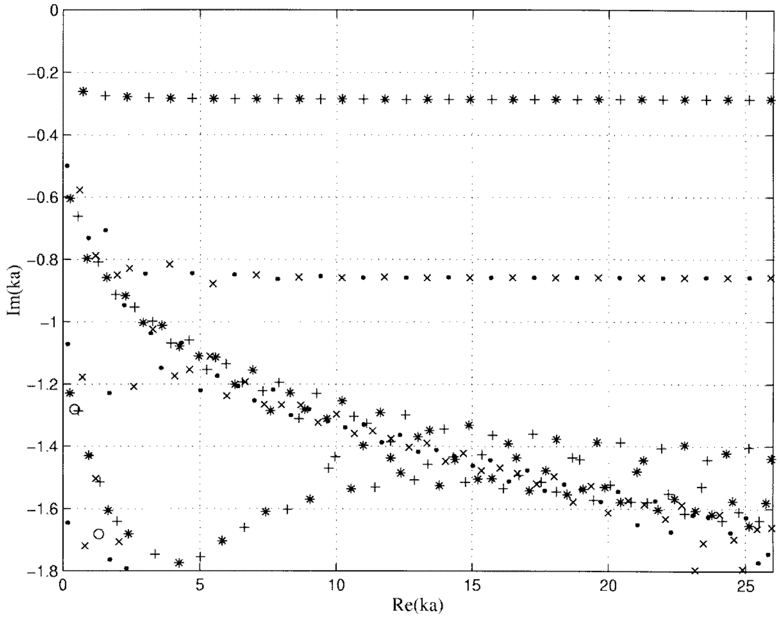

Figure 1. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{2v}$, Dirichlet boundary condition, separation distance $d = 6a$.) Resonances corresponding to one single disc are represented by open circles (○). Resonances of the representations A_1, A_2, B_1 and B_2 are respectively denoted by ($*$), ($\bullet$), ($\times$) and ($+$).

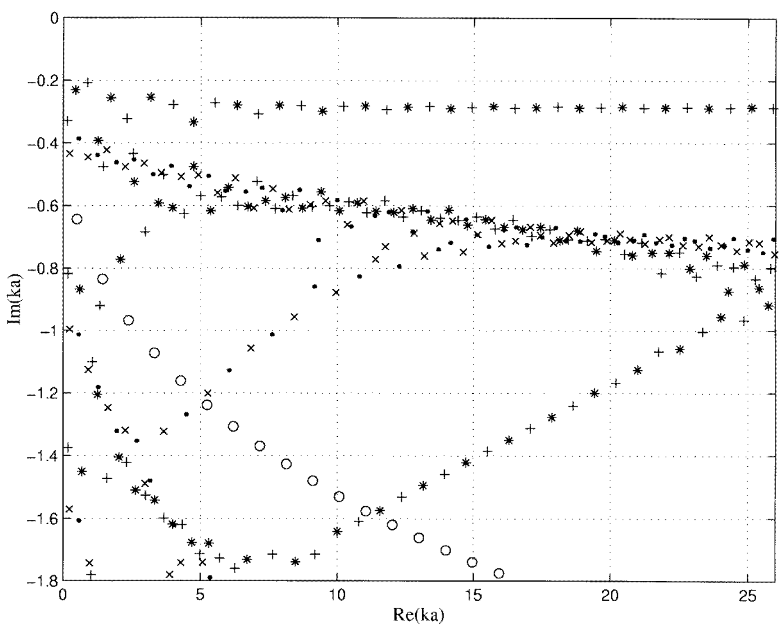

Figure 2. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{2v}$, Neumann boundary condition, separation distance $d = 6a$.) Resonances corresponding to one single disc are represented by open circles (○). Resonances of the representations A_1, A_2, B_1 and B_2 are respectively denoted by ($*$), ($\bullet$), ($\times$) and ($+$).

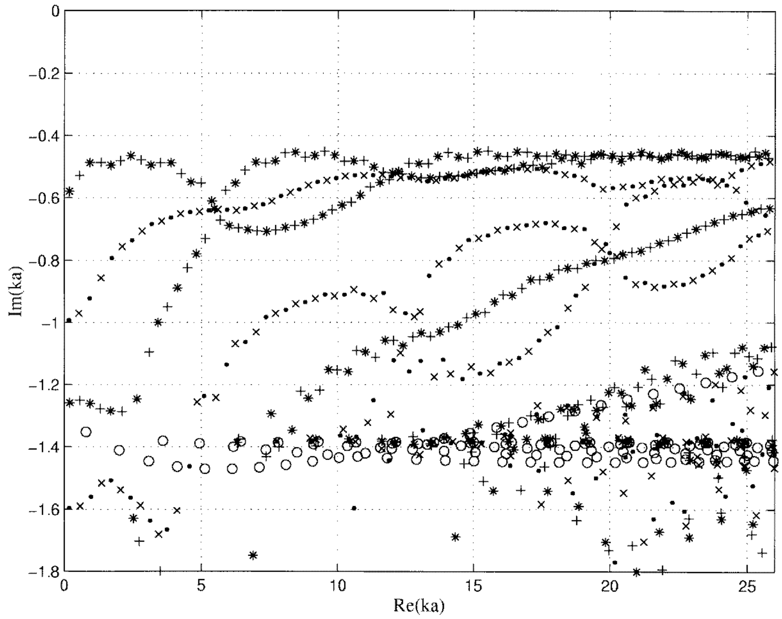

Figure 3. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{2v}$, mixed boundary conditions for $n = 1.1$, separation distance $d = 6a$.) Resonances corresponding to one single disc are represented by open circles (○). Resonances of the representations A_1, A_2, B_1 and B_2 are respectively denoted by ($*$), ($\bullet$), ($\times$) and ($+$).

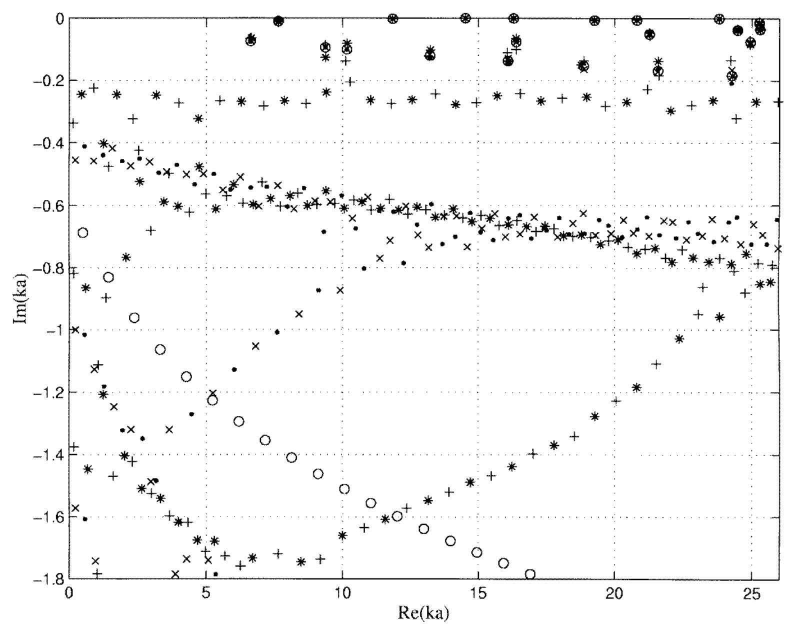

Figure 4. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{2v}$, elastic boundary conditions for two tungsten carbide cylinders immersed in water, separation distance $d = 6a$.) Resonances corresponding to one single cylinder are represented by open circles (○). Resonances of the representations A_1, A_2, B_1 and B_2 are respectively denoted by ($*$), ($\bullet$), ($\times$) and ($+$).

Resonances are distributed along certain curves in the complex ka-plane. A physical interpretation of these different curves can be obtained using semiclassical approximations in quantum mechanics or high-frequency approximations in acoustics and electromagnetism. The two-disc problem for Dirichlet boundary condition has already been treated by Vattay, Wirzba and Rosenqvist [2–4] by including, in the Gutzwiller trace formula, diffractive periodic orbits due to creeping waves. The case of Neumann boundary condition could be treated by changing slightly their formalism. In contrast, in the context of the more general physical problems examined in this paper, the construction of a periodic orbit theory seems to be a formidable task. Indeed, the existence of transmitted contributions inside the discs and, in addition to Franz waves [5], the presence of new families of surface waves (due to the coupling between the discs and the external media) significantly complicate that problem.

In the case of elastic boundary conditions, it should be noted that far from the real ka-axis the distribution of resonances is very close to the distribution corresponding to a Neumann boundary condition. In contrast, for $-0.4 \lesssim \mathrm{Im}\,(ka) \lesssim 0$, new features occur. All the resonances generated on a single isolated cylinder by the Rayleigh surface wave and the whispering gallery surface waves disappear because of the interactions between the two cylinders. They are split up into four new resonances, each one corresponding to an irreducible representation of $\mathcal{C}_{2v}$.

We have partially classified the resonances of the two-disc scatterer. They lie in four distinct families associated with the four irreducible representations A_1, A_2, B_1 and B_2 of the symmetry group of the scatterer. It would be interesting to obtain a full classification of the two-disc scatterer resonances. With this aim in view, algebraic topology which provides a classification of paths in terms of homotopy groups could be useful. Indeed, this full classification should be surely based on the free product of $\mathbb{Z}$ by $\mathbb{Z}$, the fundamental group of the plane with two disjoint holes [6].

2.2. Symmetry $\mathcal{C}_{3v}$.

We present in figures 5–8 the locations of the resonances for Dirichlet, Neumann, mixed and elastic boundary conditions. In all these cases, the centre-to-centre separation $d = 6a$. In the case of mixed boundary conditions, the refraction index $n = 0.8$. In the case of elastic boundary conditions, tungsten carbide cylinders immersed in water are considered. The splitting up of resonances occurring in the case of elastic boundary conditions is detailed in figure 9.

Cross sections and locations of resonances are shown in figures 10–13 for a small interdisc distance $d = 2.15a$. Dirichlet, Neumann and mixed ($n = 1.33$) boundary conditions are considered. The smooth variations of the averaged total cross section with ka are due to interferences between the waves scattered by the discs. Furthermore, rapid variations of sharp characteristic shape can be observed. They correspond to complex resonances near the real ka-axis and can be associated with the geometrical periodic orbit of figure 14. For a Dirichlet boundary condition, such sharp characteristic shapes have also been observed and interpreted by Gaspard and Rice [7]. In the case of mixed boundary conditions, the splitting up of resonances can be directly observed on the averaged total cross section (see figures 12 and 13).

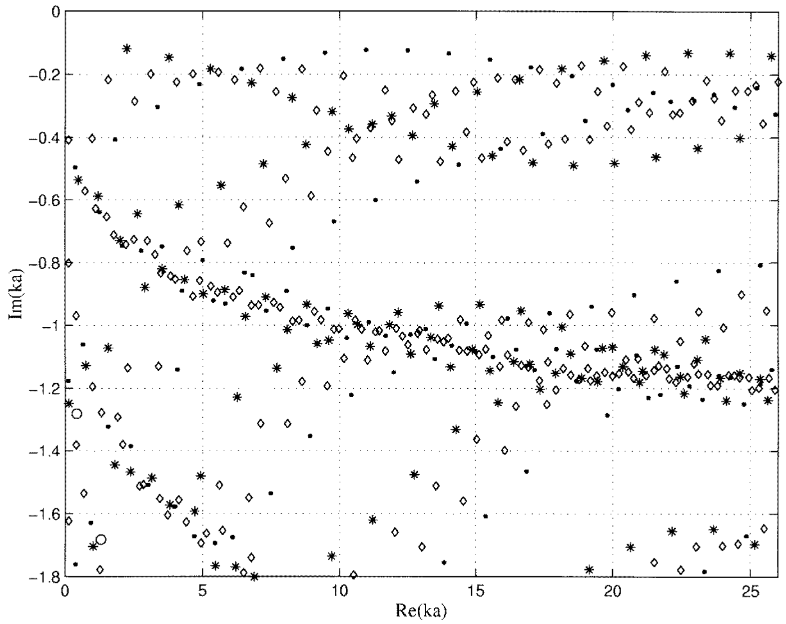

Figure 5. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{3v}$, Dirichlet boundary condition, separation distance $d = 6a$.) Resonances corresponding to one single disc are represented by open circles (○). Resonances of the representations A_1, A_2 and E are respectively denoted by ($*$), ($\bullet$) and ($\Diamond$).

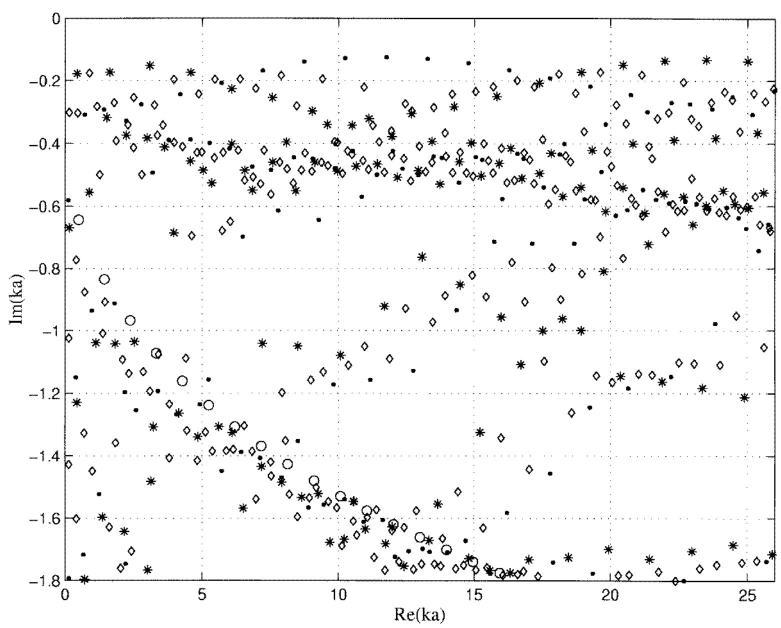

Figure 6. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{3v}$, Neumann boundary condition, separation distance $d = 6a$.) Resonances corresponding to one single disc are represented by open circles (○). Resonances of the representations A_1, A_2 and E are respectively denoted by ($*$), ($\bullet$) and ($\Diamond$).

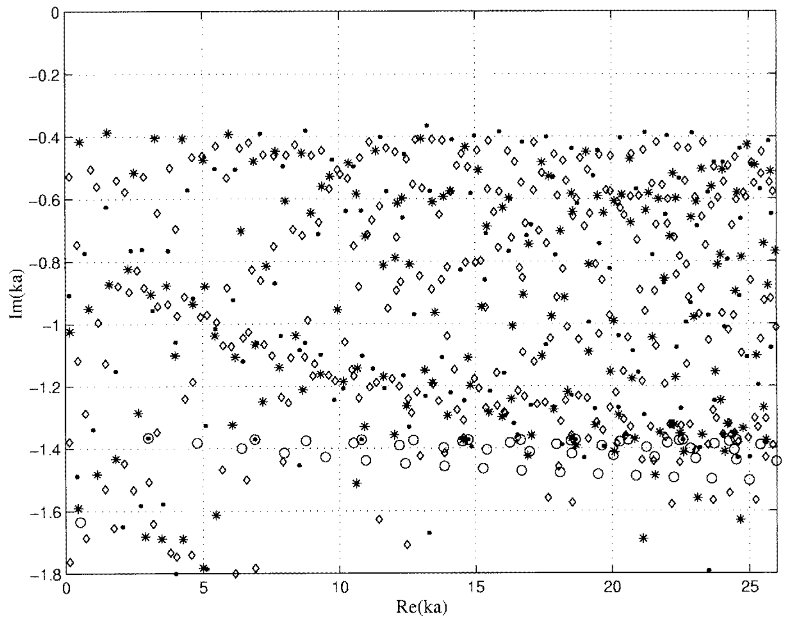

Figure 7. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{3v}$, mixed boundary conditions for $n = 0.8$, separation distance $d = 6a$.) Resonances corresponding to one single disc are represented by open circles (○). Resonances of the representations A_1, A_2 and E are respectively denoted by ($*$), ($\bullet$) and ($\Diamond$).

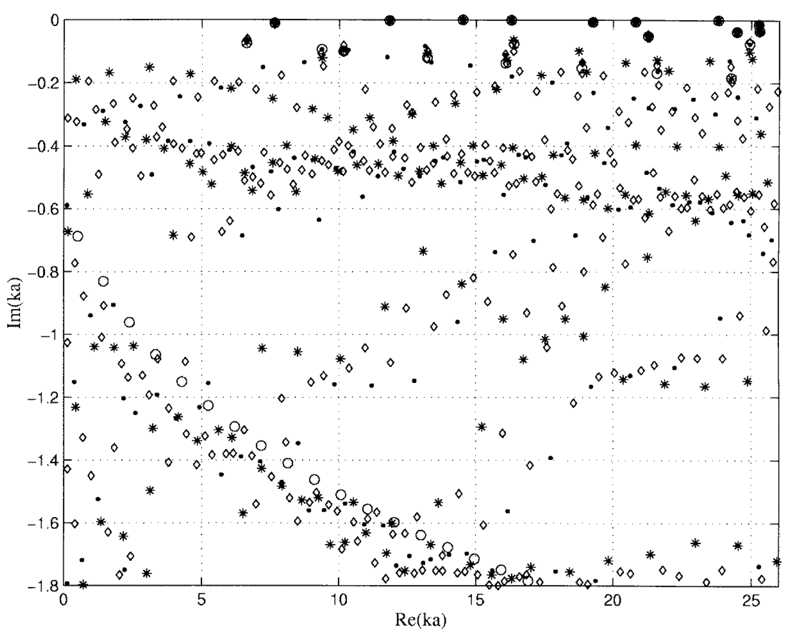

Figure 8. Location of the scattering resonances in the complex ka-plane. (Symmetry $\mathcal{C}_{3v}$, elastic boundary conditions for three tungsten carbide cylinders immersed in water, separation distance $d = 6a$.) Resonances corresponding to one single cylinder are represented by open circles (○). Resonances of the representations A_1, A_2 and E are respectively denoted by ($*$), ($\bullet$) and ($\Diamond$).

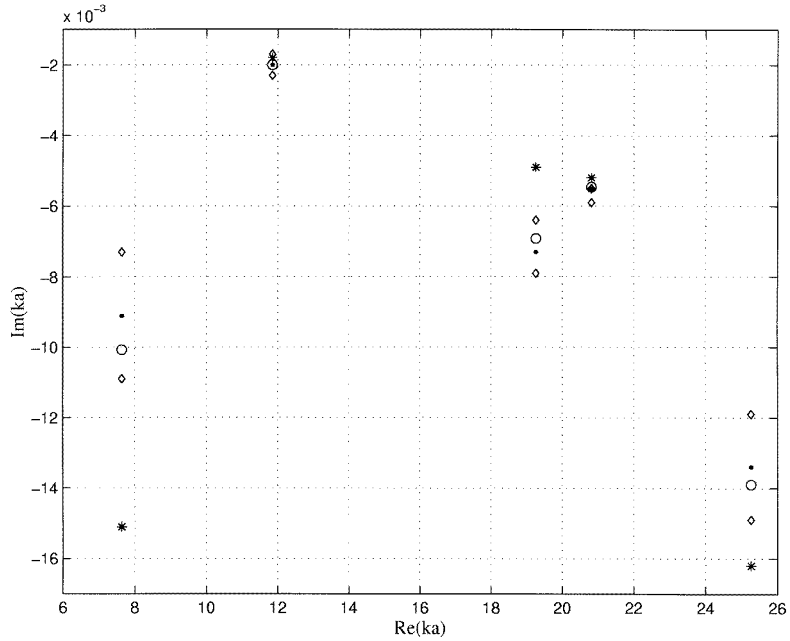

Figure 9. Zoom in on figure 8 in the domain Re $(ka) \in [6, 26]$ and Im $(ka) \in [-0.016, -0.001]$. Splitting up of resonances.

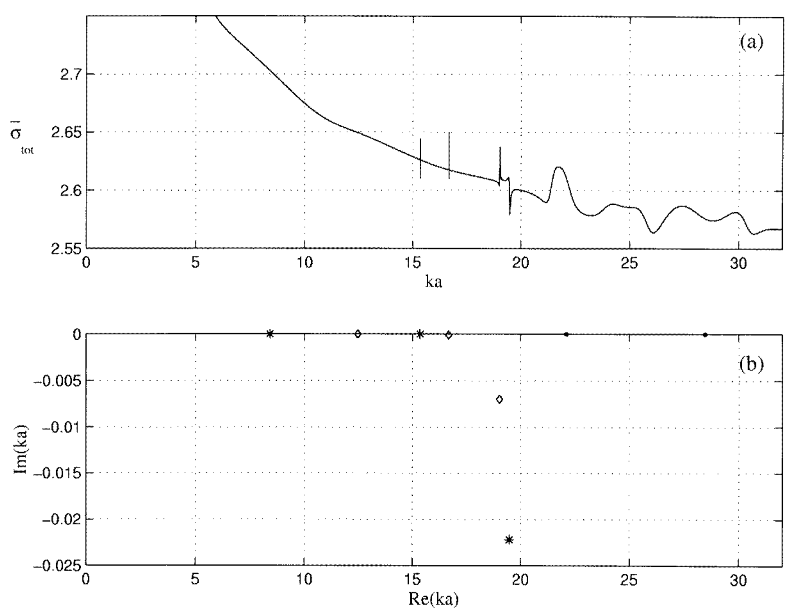

Figure 10. Symmetry $\mathcal{C}_{3v}$, Dirichlet boundary condition, separation distance $d = 2.15a$. (a) Averaged total scattering cross section $\overline{\sigma}_{tot}$. (b) Corresponding scattering resonances in the complex ka-plane.

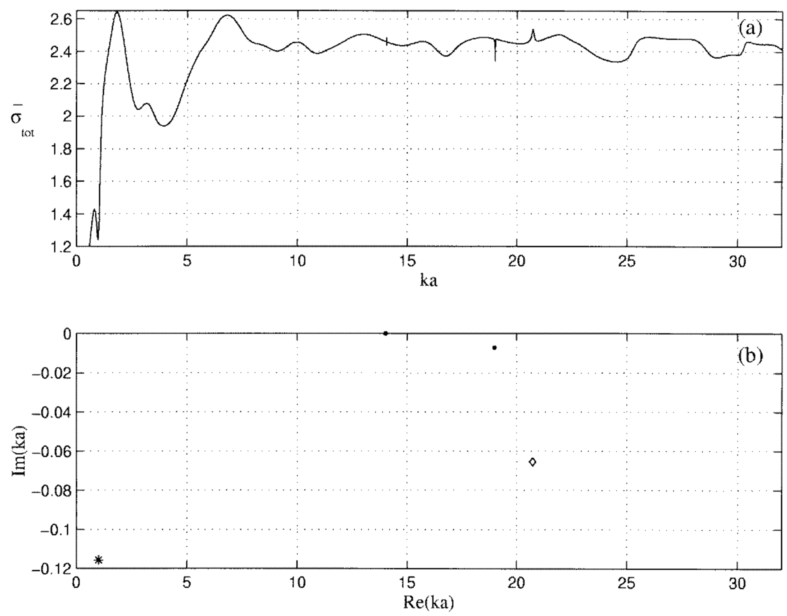

Figure 11. Symmetry $\mathcal{C}_{3v}$, Neumann boundary condition, separation distance $d = 2.15a$. (*a*) Averaged total scattering cross section $\overline{\sigma}_{tot}$. (*b*) Corresponding scattering resonances in the complex ka-plane.

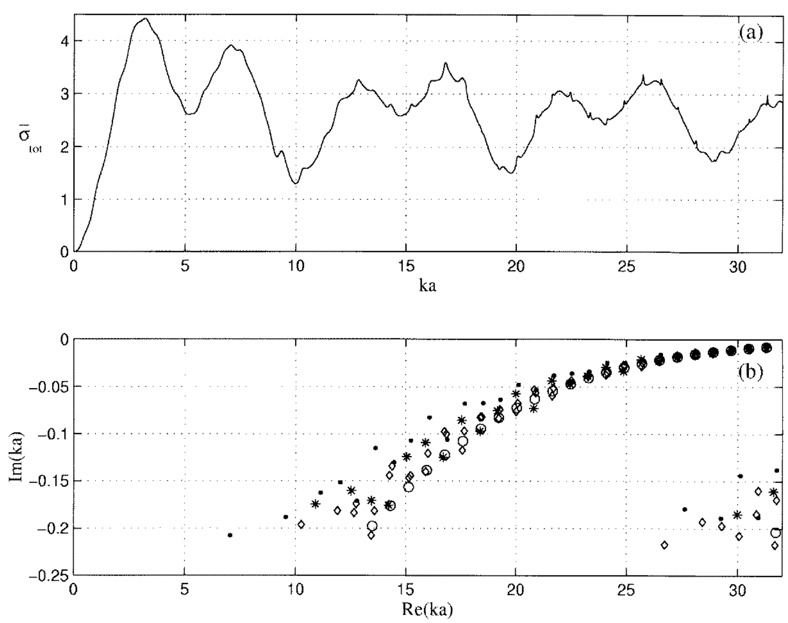

Figure 12. Symmetry $\mathcal{C}_{3v}$, mixed boundary conditions for $n = 1.33$, separation distance $d = 2.15a$. (*a*) Averaged total scattering cross section $\overline{\sigma}_{tot}$. (*b*) Location of the scattering resonances in the complex ka-plane.

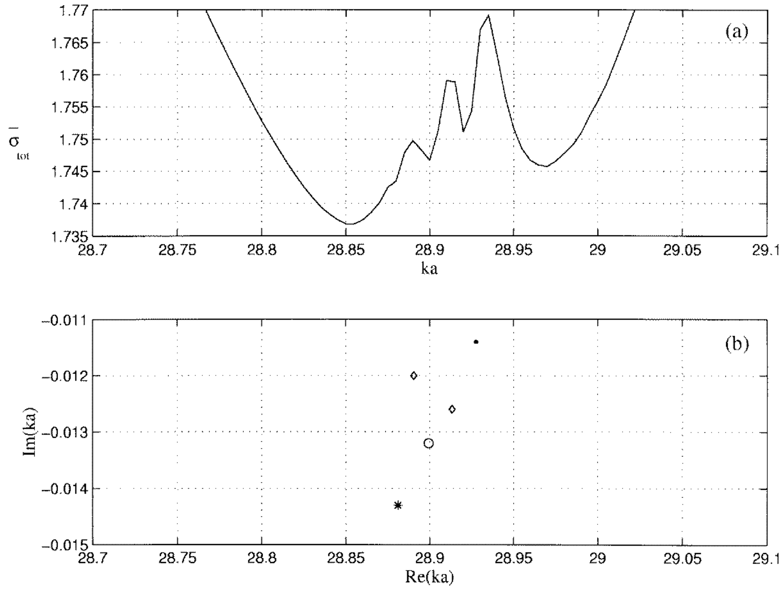

Figure 13. Zoom in on figure 12. (*a*) Averaged total scattering cross section $\overline{\sigma}_{tot}$ in the domain $ka \in [28.7, 29.1]$ and $\overline{\sigma}_{tot} \in [1.735, 1.770]$. (*b*) Location of the scattering resonances in the complex ka-plane in the domain $\mathrm{Re}\,(ka) \in [28.7, 29.1]$ and $\mathrm{Im}\,(ka) \in [-0.015, -0.011]$. Splitting up of resonances.

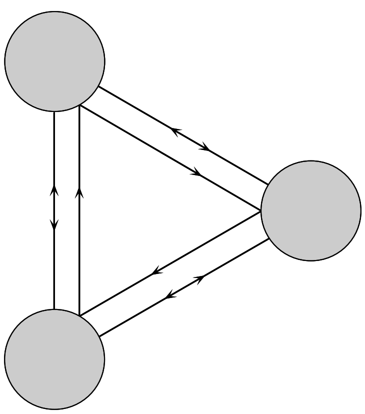

Figure 14. An example of geometrical periodic orbit trapped in the three-disc system.

3. Conclusion and perspectives

In this paper, we have determined the scattering resonances of the two- and three-disc systems with discrete $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ symmetries. It would be interesting to experimentally confirm the expected physical effects such as, for example, the splitting up of resonances and resonances of interaction between the diffusors. In electromagnetism, in the context of microwave two-disc scattering, Kudrolli and Sridhar [8] observed resonances corresponding to the A_2 antisymmetric poles of the S-matrix. In acoustics, similar experiments in the case of the three-disc scatterer are in preparation [9].

Furthermore, we would like to link our exact results with those obtained from semiclassical approximations in quantum mechanics or high-frequency approximations in acoustics and electromagnetism. In the context of the three-hard-disc system, Gaspard and Rice have studied the scattering of a point particle in the semiclassical approximation using the Gutzwiller trace formula [10]. More recently, Vattay, Wirzba and Rosenqvist [3, 4] (see also [2]) extend the Gutzwiller trace formula including diffractive periodic orbits due to creeping waves. Their approach provides a good agreement between locations of exact quantum mechanical resonances and locations of their semiclassical approximations. Unfortunately, in the context of the physical problems examined in this paper, a periodic orbit theory cannot be easily developed because of the existence of transmitted contributions inside the discs and, in addition to Franz waves, the presence of new families of surface waves. However, with this aim in view, the formalism developed in [11] in the context of semiclassical quantization of billiards with mixed boundary conditions could be useful.

Finally, it would be interesting to consider the statistical characteristics of the S-matrix previously obtained in order to emphasize the chaotic aspects of multiple scattering. The random matrix description exploited by Fyodorov and Sommers [12] (see also references therein) could be applied in the three-disc problem for $ka \gg k(d-2a) \gg 1$. Indeed, in this case, the 'internal' and the 'external' parts of the scatterers are well defined and the numbers of channels relating these two parts is small.

Acknowledgment

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References

[1] Decanini Y, Folacci A, Fournier E and Gabrielli P 1998 Exact S-matrix for N-disc systems and various boundary conditions: I. Generalization of the Korringa–Kohn–Rostoker–Berry method *J. Phys. A: Math. Gen.* **31** 7865

[2] Wirzba A 1992 Validity of the semiclassical periodic orbit approximation in the two- and three-disk problems *Chaos* **2** 77–83

[3] Vattay G, Wirzba A and Rosenqvist P E 1994 Periodic orbit theory of diffraction *Phys. Rev. Lett.* **73** 2304–7

[4] Rosenqvist P E, Vattay G and Wirzba A 1995 Application of the diffraction trace formula to the three-disk scattering system *J. Stat. Phys.* **83** 243–57

[5] Franz W 1957 *Theorie der Beugung Elektromagnetischer Wellen* (Berlin: Springer)

[6] Nash C and Sen S 1983 *Topology and Geometry for Physicists* (London: Academic)

[7] Gaspard P and Rice S A 1989 Exact quantization of the scattering from a classically chaotic repellor *J. Chem. Phys.* **90** 2255–62

[8] Kudrolli A and Sridhar S 1995 *Proc. 4th Drexel Conf.* (Singapore: World Scientific) to appear

[9] Decanini Y, Folacci A, Fournier E and Gabrielli P 1998 in preparation

[10] Gaspard P and Rice S A 1989 Semiclassical quantization of the scattering from a classically chaotic repellor *J. Chem. Phys.* **90** 2242–54

[11] Sieber M, Primack H, Smilansky U, Ussishkin I and Schanz H 1995 Semiclassical quantization of billiards with mixed boundary conditions *J. Phys. A: Math. Gen.* **28** 5041–78

[12] Fyodorov Y V and Sommers H J 1997 Statistics of resonances poles, phase shifts and time delays in quantum chaotic scattering: random matrix approach for systems with broken time-reversal invariance *J. Math. Phys.* **38** 1918–81